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Solar Sail Equilibria in the Elliptical Restricted Three-Body Problem

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The existence and dynamical properties of artificial equilibria for solar sails in the elliptical restricted three-body problem is investigated. We show that planar two-dimensional equilibrium curves exist, embedded in three-dimensional space, in a nonuniformly rotating, pulsating coordinate system. However, due to the stretching of the system plane coordinates and unstretching of the out-of-plane coordinate, the equilibrium surfaces do not exist in the three-dimensional elliptical restricted three-body system. Control in the neighborhood of an equilibrium point is investigated through a pole assignment scheme. This permits practical out-of-plane equilibria in elliptical three-body systems with small eccentricity.

Nomenclature

$\boldsymbol{a}$	=	solar pressure force acting on the sail
f	=	true anomaly of the smaller primary
N	=	complex number notation of the sail normal vector
$\boldsymbol{n}$	=	sail normal vector
r_1, r_2	=	distances from sail to the larger and smaller primaries, respectively (in pulsating coordinates)
$\tilde{r}$	=	distance between two primaries
$\tilde{r}_1, \tilde{r}_2$	=	distances from sail to the larger and smaller primaries, respectively (in nonpulsating coordinates)
$\boldsymbol{s}$	=	sail position vector in pulsating coordinates
X, Y	=	initial coordinate fixed at system mass center
x, y, z	=	rotating and pulsating coordinate
W	=	complex number notation of the initial coordinate
w	=	complex number notation of the rotating, pulsating coordinate
α	=	angle between sail normal vector and sunlight
β	=	sail lightness number
μ	=	dimensionless mass of the smaller primary
ξ, η	=	nonuniformly rotating coordinate
ς	=	complex number notation of the nonuniformly rotating coordinate
Ω	=	pseudopotential function

Introduction

DYNAMICS and control of solar sails in the circular restricted three-body problem have been investigated in some detail.[1–3] The problem (aluminous primary mass and a flat mirror) possesses infinite equilibrium surfaces parameterized by the solar sail lightness number, whereas the classical elliptical restricted three-body problem possesses only five Lagrange points.[4] The so-called photogravitational circular restricted problem (aluminous primary mass and a spherical test particle) has seven Lagrange points.[5] The stability of these seven Lagrange points have been investigated by several authors for either the circular or elliptical restricted case.[6–11] In those papers, instead of "solar sail," the term "photogravitational" is adopted and is equivalent to the solar sail problem with the sail normal fixed along the sun line (spherical test particle). It can be shown that the collinear equilibrium points of the photogravitational elliptical restricted three-body problem are unstable, whereas the triangular points are Lyapunov stable in some parameter ranges.

Using the solar sail circular restricted three-body problem, various applications of the artificial equilibria have been discussed. Forward[12] proposed using artificial equilibria high above the night side of the Earth (equilibrium surfaces attached to the L_2 point) to provide telecommunications services to high-latitude users, whereas McInnes[13,14] has proposed using artificial equilibria high above the day side of the Earth (equilibrium surfaces attached to the L_1 point) to provide continuous, real-time imaging of the poles. Morrow et al. also proposed using three-body equilibrium surfaces to park solar sails in close proximity to asteroids or other small solar system bodies.[15] In-plane equilibria have also been considered as a useful location at which to station space weather missions, sunward of the classical L_1 point.[16]

In this article the solar sail elliptical restricted three-body problem is investigated in some detail. The elliptical restricted three-body problem assumes that two point mass primary bodies revolve around their mass center in elliptical orbits under the influence of their mutual gravitational attraction. The third body moves under the influence of the two primaries, and its mass is negligible. The planar case is the motion of the third body restricted to the two primary's orbit plane.

First, a dimensionless equation of motion is derived using a nonuniformly rotating pulsating coordinate system and the existence of planar equilibrium curves is investigated. We then show that the three-dimensional equilibrium surfaces, which exist in the circular restricted three-body problem, do not exist for the elliptical restricted three-body problem. This is due to the stretching of the coordinate frame in the system plane, whereas the out-of-plane coordinate remains unstretched. Instead, it is shown that planar two-dimensional equilibrium curves exist, embedded in three-dimensional space, in the nonuniformly rotating, pulsating coordinate system. The local stability of these equilibria is studied numerically and it is demonstrated that, in general, the equilibria are unstable. Finally, a control scheme for the stabilization of out-of-plane equilibria will be derived for the small eccentricity case. This analysis demonstrates the possibility of practical out-of-plane equilibria in small eccentricity systems, even though exact out-of-plane equilibria do not exist for the general elliptical restricted three-body problem. This is a key finding because out-of-plane equilibria in the Earth–sun three-body system have been proposed for near-term applications of solar sailing.[12–14]

Equations of Motion

Consider a planar elliptical restricted three-body system. Assume that an appropriate set of units is introduced so that the gravitational constant $G = 1$, the semimajor axis of the problem equals 1, and

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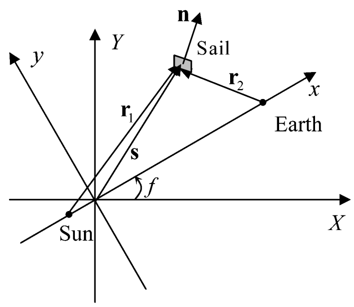

Fig. 1 Nonuniformly rotating and pulsating coordinate system.

the system has unit total mass. Let μ be the dimensionless mass of the smaller primary (Earth) and then $1-\mu$ will be the mass of the larger primary (sun), which is luminous. As shown in Fig. 1, let X, Y denote a fixed inertial coordinate frame, and using complex number notation define

$$W = X + iY \tag{1}$$

Then, in a similar manner, let ξ, η denote a nonuniformly rotating coordinate frame, where

$$\varsigma = \xi + i\eta \tag{2}$$

Again, using complex number notation, the two reference frames can be connected through a rotation

$$W = \varsigma e^{if} \tag{3}$$

where f is the true anomaly of the smaller primary in the elliptical three-body system, as shown in Fig. 1. We use nonsubscripted variables to denote the sail coordinate hereafter.

Using this notation, the equation of motion of a solar sail in the inertial coordinate system can be written by summing the gravitational acceleration of the two primary masses and the solar radiation pressure acceleration experienced by the solar sail to obtain

$$\ddot{W} = -(1-\mu)\left[(W-W_1)\big/\tilde{r}_1^3\right] - \mu\left[(W-W_2)\big/\tilde{r}_2^3\right] + \beta(1-\mu)\left(\cos^2\alpha\big/\tilde{r}_1^2\right)N \tag{4}$$

where α is the angle between the sail normal and the sunlight, N is the sail unit normal vector, and W_1, W_2 are coordinates of the sun and Earth, respectively, whereas

$$\tilde{r}_1 = \sqrt{(\xi+\mu)^2+\eta^2}, \qquad \tilde{r}_2 = \sqrt{(\xi+\mu-1)^2+\eta^2}$$

are the distances from the two primary masses to the solar sail. Again using complex notation, the sail unit normal vector can be written as

$$N = N_X + iN_Y$$

whereas the sail lightness number β is defined as the ratio of the solar radiation pressure acceleration to the solar gravitational acceleration.[1] From Eq. (3) the second time derivative of W can be obtained as

$$\ddot{W} = [\ddot{\varsigma} + 2i\dot{f}\dot{\varsigma} - \dot{f}^2\varsigma + i\ddot{f}\varsigma]e^{if} \tag{5}$$

Then, substituting Eq. (5) into Eq. (4), the equation of motion of the solar sail in the rotating coordinate frame can be written as

$$\ddot{\varsigma} + 2i\dot{f}\dot{\varsigma} = -(1-\mu)\left[(\varsigma-\varsigma_1)\big/\tilde{r}_1^3\right] - \mu\left[(\varsigma-\varsigma_2)\big/\tilde{r}_2^3\right] + \beta(1-\mu)\left(\cos^2\alpha\big/\tilde{r}_1^2\right)n + \varsigma\dot{f}^2 - i\ddot{f}\varsigma \tag{6}$$

where $N = ne^{if}$ with n now defined in the rotating frame, and $W_1 = \varsigma_1 e^{if}$, $W_2 = \varsigma_2 e^{if}$. In addition, a nondimensional (complex) position coordinate can be defined as

$$w = \varsigma/\tilde{r} \tag{7}$$

where $\tilde{r}$ is the distance between the two primaries, which can be obtained from the two-body problem, and

$$w = x + iy \tag{8}$$

where x and y are the nondimensional position variables of the solar sail in the rotating and pulsating frame of reference. Because the distance between the two primaries is time-varying, the rotating coordinate frame is now pulsating (stretched in a periodic manner). Using the pulsating coordinate system it can then be shown that

$$\dot{\varsigma} = \tilde{r}\dot{w} + w\dot{\tilde{r}} \tag{9a}$$

$$\ddot{\varsigma} = \tilde{r}\ddot{w} + 2\dot{\tilde{r}}\dot{w} + w\ddot{\tilde{r}} \tag{9b}$$

and so by substituting Eqs. (7–9) into Eq. (6) it is found that

$$\tilde{r}\dot{f}^2(w''+2iw') + w(\ddot{\tilde{r}} - \tilde{r}\dot{f}^2) + (w'+iw)(\tilde{r}\ddot{f} + 2\dot{\tilde{r}}\dot{f}) = -(1-\mu)\left[(w-w_1)\big/\tilde{r}^2r_1^3\right] - \mu\left[(w-w_2)\big/\tilde{r}^2r_2^3\right] + \beta(1-\mu)\left(\cos^2\alpha\big/\tilde{r}^2r_1^2\right)n \tag{10}$$

where $r_1 = \tilde{r}_1/\tilde{r}$, $r_2 = \tilde{r}_2/\tilde{r}$ and the prime indicates derivatives with respect to the true anomaly f. Considering the two-body orbit of the primaries to determine the time derivative terms yields[4]

$$w'' + 2iw' = [1/(1+e\cos f)]\left\{w - (1-\mu)\left[(w-w_1)\big/r_1^3\right] - \mu\left[(w-w_2)\big/r_2^3\right] + \beta(1-\mu)\left(\cos^2\alpha\big/r_1^2\right)n\right\} \tag{11}$$

which can be rewritten in real number form as

$$(x''-2y') = \frac{1}{(1+e\cos f)}\left[\frac{\partial\Omega}{\partial x} + \beta(1-\mu)\frac{\cos^2\alpha}{r_1^2}n_x\right] \tag{12a}$$

$$(y''+2x') = \frac{1}{(1+e\cos f)}\left[\frac{\partial\Omega}{\partial y} + \beta(1-\mu)\frac{\cos^2\alpha}{r_1^2}n_y\right] \tag{12b}$$

where the pseudopotential of the problem can be written as

$$\Omega = \tfrac{1}{2}(x^2+y^2) + (1-\mu)/r_1 + \mu/r_2 \tag{13}$$

Equation (12) can also be written in vector form as

$$\boldsymbol{s}'' + S\boldsymbol{s}' = [1/(1+e\cos f)][\nabla\Omega + \boldsymbol{a}] \tag{14}$$

where $\boldsymbol{s} = x\boldsymbol{i} + y\boldsymbol{j}$ for unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ along the x and y axes in the rotating coordinate system

$$S = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

and

$$\boldsymbol{a} = \beta\left[(1-\mu)\big/r_1^2\right]\left[(\boldsymbol{r}_1\cdot\boldsymbol{n})^2\big/r_1^2\right]\boldsymbol{n} \tag{15}$$

where the sail unit normal vector is now written as $\boldsymbol{n} = n_x\boldsymbol{i} + n_y\boldsymbol{j}$. The equation of motion in the nonuniformly rotating, pulsating coordinate system is now used to investigate the existence of equilibria in the solar sail elliptical restricted three-body problem.

Equilibrium Solutions

To determine whether equilibria exist in the solar sail elliptical restricted three-body problem, the equilibrium conditions $x'' = y'' = x' = y' = 0$ are substituted into Eq. (14). Thus, equilibrium solutions must satisfy the identity

$$\nabla\Omega + \boldsymbol{a} = 0 \tag{16}$$

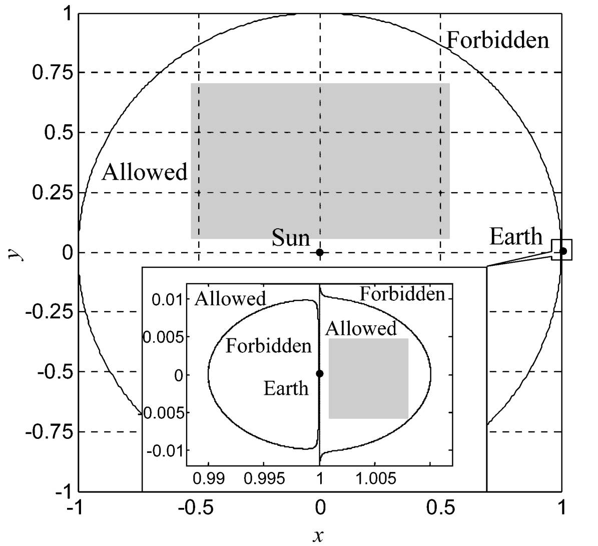

Fig. 2 Region of existence of equilibrium solutions and stability areas analyzed (shaded areas).

Taking the vector product of $\boldsymbol{n}$ with Eq. (16) yields Eq. (17a), and the scalar product of that follows as Eq. (17b):

$$\nabla\Omega \times \boldsymbol{n} = 0 \Rightarrow \boldsymbol{n} = -\frac{\nabla\Omega}{|\nabla\Omega|} \tag{17a}$$

$$\beta = -\frac{1}{(1-\mu)} \frac{(\nabla\Omega \cdot \boldsymbol{n}) r_1^4}{(\boldsymbol{r}_1 \cdot \boldsymbol{n})^2} \tag{17b}$$

Equation (17a) implies that at an equilibrium point the sail normal is always directed opposite to the sum of the gravitational force from the primary masses and the centripetal force acting on the solar sail, whereas Eq. (17b) can be used to determine the required sail lightness number for equilibrium. In addition, because the condition for equilibrium for the elliptical case (in the rotating, pulsating coordinate frame) has a form similar to that in the circular case,[1,2] the planar equilibrium curves in the nonuniformly rotating, pulsating coordinate frame are similar to the equilibrium curves in the system plane of the circular case discussed in Refs. 1 and 2. Far from the Earth the planar equilibrium curves are nearly circular. However, there is a more complex topology near the Earth, with allowed and forbidden areas, as shown in Fig. 2. Because the coordinate systems are different in the elliptical and circular cases, there is a significant difference between the two families of equilibrium curves. Whereas the planar equilibrium solution curves of the circular case are invariant in a uniformly rotating coordinate frame, the curves of the elliptical case are deforming as the primaries orbit each other, just as the distance between the classical Lagrange points and the primaries varies in the elliptical restricted three-body problem.[4]

Figure 3 shows the equilibrium curves in a nonpulsing coordinate frame for Earth, where $f = 0$, for the far Earth case (Fig. 3a) and the near-Earth case (Fig. 3b). To demonstrate the effects of large eccentricity on the size and position changes of equilibrium curves, Fig. 4 shows the equilibrium curves near Mercury, where $f = 0$ (Fig. 4a), $f = \pi/2$ (Fig. 4b), and $f = \pi$ (Fig. 4c). Because of the large eccentricity of Mercury, the curves change significantly both in position and in size. (Note that here the dimensionless distance is based on the semimajor axis of Earth's orbit.) System parameters are taken as $\mu = 3.04036 \times 10^{-6}$, $e = 0.0167$ for Earth, and $\mu = 1.672 \times 10^{-7}$, $e = 0.206$ for Mercury. The figures indicate that the position and size of the equilibrium curves are varying in a nonpulsing coordinate frame, although they are not varying in the pulsing coordinate frame.

Finally, the three-dimensional case is considered. Using the preceding analysis, it can be shown that the dimensionless equation of

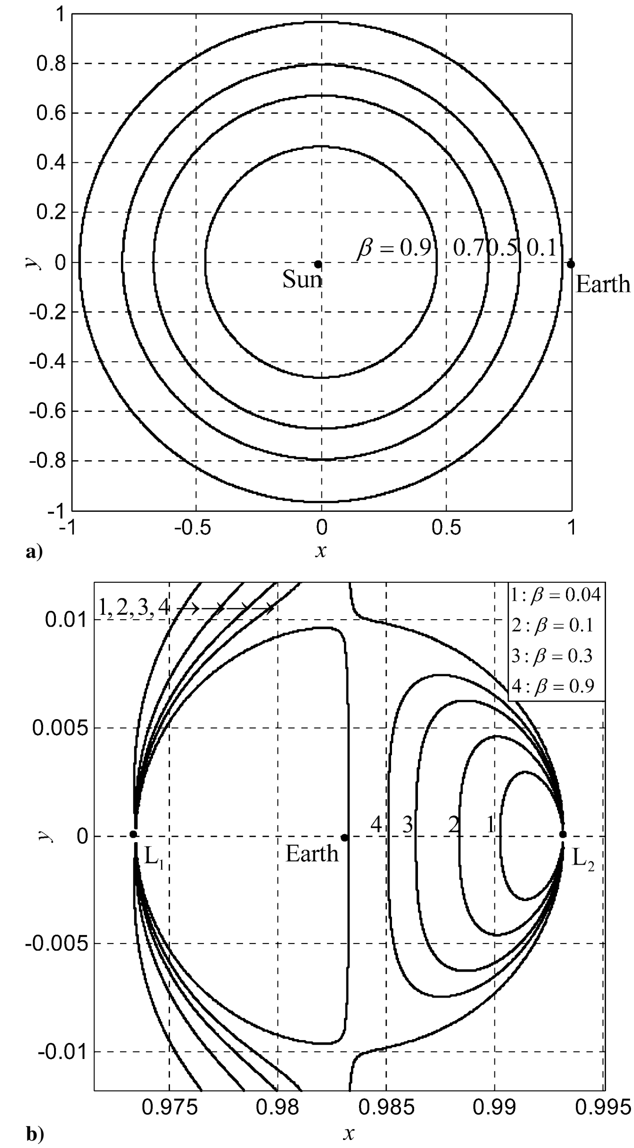

Fig. 3 Equilibrium solution curves for Earth ($f = 0$): a) far from Earth and b) near Earth.

the motion can be written as

$$(x'' - 2y') = \frac{1}{(1 + e\cos f)}\left[\frac{\partial\Omega}{\partial x} + \beta(1-\mu)\frac{\cos^2\alpha}{r_1^2} n_x\right] \tag{18a}$$

$$(y'' + 2x') = \frac{1}{(1 + e\cos f)}\left[\frac{\partial\Omega}{\partial y} + \beta(1-\mu)\frac{\cos^2\alpha}{r_1^2} n_y\right] \tag{18b}$$

$$(z'' + z) = \frac{1}{(1 + e\cos f)}\left[\frac{\partial\Omega}{\partial z} + \beta(1-\mu)\frac{\cos^2\alpha}{r_1^2} n_z\right] \tag{18c}$$

where

$$\Omega = \tfrac{1}{2}(x^2 + y^2 + z^2) + (1-\mu)/r_1 + (\mu/r_2)$$

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}$$

To obtain an equilibrium point, Eq. (18) must be solved under the condition $x'' = x' = y'' = y' = z'' = 0$. It yields a group of algebraic equations essentially including the time-varying term $(1 + e\cos f)z$; therefore the solution of the algebraic equations is expected to be a function of the variable f,which conflicts with the

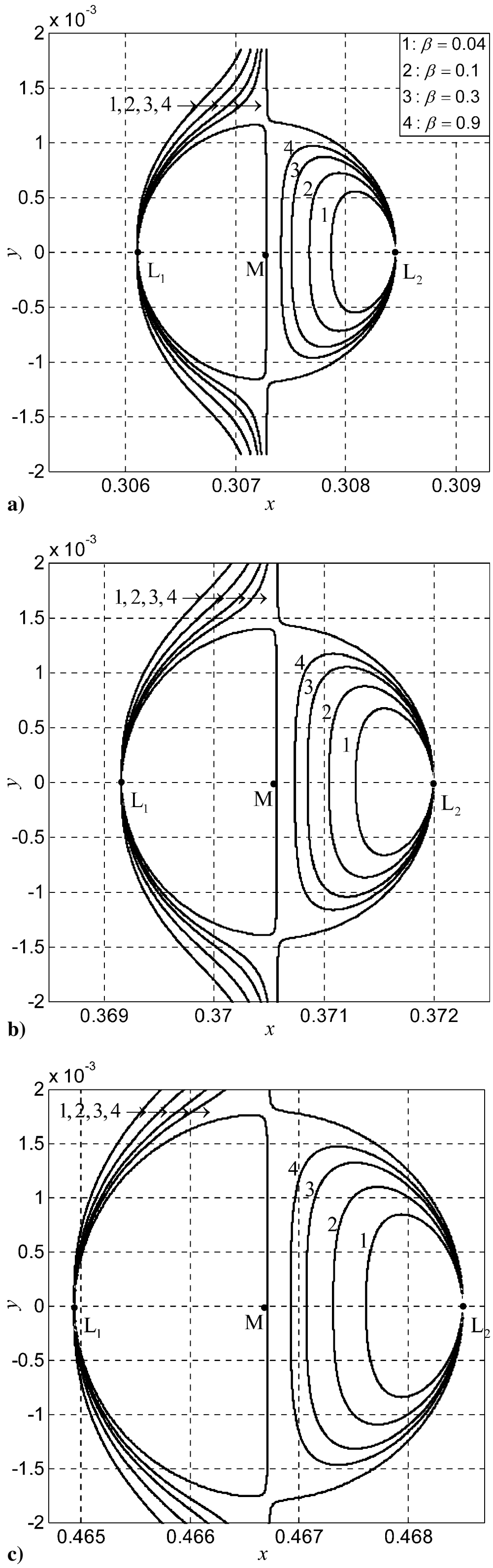

Fig. 4 Equilibrium solution curves in the vicinity of Mercury: a) $f = 0$, b) $f = \pi/2$, and c) $f = \pi$.

condition aforementioned. It implies that Eq. (18) does not possess the equilibrium solution and there are no three-dimensional equilibrium surfaces in the non-uniformly rotating, pulsating coordinate frame. Physically, due to the periodic distance change between the two primaries, the time-dependent equilibrium points do exist in the system plane; but due to the different periodic nature of the motion between the system plane and its vertical direction, the out-of-plane equilibrium points do not exist in the three-dimensional case. Therefore, for the solar sail elliptical restricted three-body problem, equilibrium solutions only exist in the plane of the system as two-dimensional planar curves embedded in three-dimensional space. This is due to the stretching of the coordinate frame in the system plane, while the out-of-plane coordinate remains unstretched.

The out-of-plane equilibria in the sun–Earth restricted three-body problem have been proposed as a near-term application of solar sailing.[12–14] The result that no out-of-plane equilibria exist appears to be problematic for this kind of mission. However, as will be seen later, active control can be used to establish practical equilibria for the small eccentricity case. Because the eccentricity of the Earth's orbit is small this ensures that applications of out-of-plane equilibria are still possible.

Stability of Equilibria

To determine the local stability of the equilibrium points in the linear sense, we apply an infinitesimal perturbation δ to an equilibrium point, and so the variation of Eq. (14) can be obtained as

$$\boldsymbol{\delta}'' + S\boldsymbol{\delta}'' - \frac{1}{(1 + e\cos f)}\left(\frac{\partial(\nabla\Omega)}{\partial \boldsymbol{s}} + \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{s}}\right)\boldsymbol{\delta} = 0 \tag{19}$$

where

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{s}} = \frac{4\nabla\Omega \boldsymbol{r}_1^T}{(\boldsymbol{r}_1 \cdot \boldsymbol{r}_1)} - \frac{2\nabla\Omega\nabla\Omega^T}{(\boldsymbol{r}_1 \cdot \nabla\Omega)} \tag{20}$$

and matrix

$$K = \left(\frac{\partial(\nabla\Omega)}{\partial \boldsymbol{s}} + \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{s}}\right)$$

is evaluated at the equilibrium point. Note that in Eq. (20) we assume that the direction of the sail normal vector remains as the infinitesimal perturbation is applied to the position variable. In addition, because matrix K is time dependent, the problem becomes the stability of a linear system with time-dependent coefficients. To study the stability, Eq. (19) can be represented in the form of first-order differential equations as

$$\boldsymbol{\chi}' = A\boldsymbol{\chi} \tag{21}$$

where $\boldsymbol{\chi} = [\boldsymbol{\delta}, \boldsymbol{\delta}']^T$ and the system matrix A is defined as

$$A = \begin{bmatrix} 0 & I \\ K/(1 + e\cos f) & -S \end{bmatrix}$$

where I is a unit matrix. Because the system has time-dependent coefficients, its stability cannot be directly determined by the eigenvalues of the matrix A. However, because $A(f) = A(f + 2\pi)$, according to Floquet theory, the stability of a periodic coefficient linear system can be determined by the system behavior over one period. We first define a matrix $Q(f)$ by

$$Q'(f) = A(f)Q(f) \tag{22}$$

where $Q(0) = I$ is an appropriate-dimensional unit matrix. Then Eq. (22) is integrated to obtain $Q(2\pi)$. If the eigenvalues λ_i of matrix $Q(2\pi)$ satisfy $|\lambda_i| \le 1$, the system is stable; otherwise it is unstable. Unfortunately, because the matrix $A(f)$ cannot commute with its integral matrix

$$\int_0^f A(\tau)\,\mathrm{d}\tau$$

we cannot integrate Eq. (22) to get the closed-form solution of $Q(2\pi)$, but numerical integration is always available.

We have investigated the stability of equilibria in some areas, defined by the shaded area of Fig. 2, using numerical integration, but only with the parameters $\mu = 3.04036 \times 10^{-6}$ and $e = 0.0167$ for the Earth–sun elliptical restricted three-body problem. Numerical results show that those equilibria in the shaded area are all unstable, as expected. Indeed, we expect all equilibria for the solar sail elliptical restricted three-body problem to be unstable, other than the classical triangular points for some parameter ranges.[7] The photogravitational problem always implicitly includes an assumption that the solar radiation pressure force is always along sunlight, but if this assumption were considered in the solar sail problem, term $\partial \boldsymbol{n}/\partial \boldsymbol{s}$ would be considered in Eq. (20). Therefore, even in the triangular equilibrium points, stability of the photogravitational and solar sail problem is not the same.

Control of Out-of-Plane Equilibria

To enable unstable equilibria to be used for practical applications, active control is required. In addition, to allow out-of-plane equilibria to be used in the small eccentricity case, active control is required. Here we consider orbit control manipulated with active control of the sail attitude.

Usually, a nonlinear system is linearized around an equilibrium point and controlled in a local neighborhood of it. However, in the solar sail elliptical restricted three-body problem it has been shown that out-of-plane equilibria do not exist. In the case $e \ll 1$, if the term $e \cos f$ is ignored in Eq. (18), we can obtain an approximate equilibrium point that is positioned in the pulsing coordinate in a manner similar to that in the circular case (because the out-of-plane axis is not pulsating).

To consider linear control around an approximated equilibrium point, we assume that Eq. (18) is linearized around the approximated equilibrium point ($e \ll 1$), and the sail attitude is used as the control variable. Thus, the controlled equation can be written as

$$\chi' = A\chi + B\boldsymbol{u} \tag{23}$$

where $\chi = [x, y, z, x', y', z']^T$ and

$$A = \begin{bmatrix} 0 & I \\ K/(1+e\cos f) - \bar{K} & -\bar{S} \end{bmatrix}, \qquad \bar{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{S} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \mathbf{0} & \dfrac{\partial \boldsymbol{a}}{\partial \boldsymbol{n}} \end{bmatrix}^T_{3\times 6} \tag{24}$$

where $\boldsymbol{n} = [n_x, n_y, n_z]^T$, $\boldsymbol{u} = \delta \boldsymbol{n}$, and

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{n}} = |\nabla\Omega| I + \frac{2|\nabla\Omega|}{(\nabla\Omega \cdot \boldsymbol{r}_1)} \nabla\Omega \boldsymbol{r}_1^T \tag{25}$$

This is a control problem of a time-varying linear system, because both A and B are time-dependent matrices. Therefore, using standard time-varying linear–quadratic theory,[17] one can design an optimal control law for the system. In what follows, however, we provide a simpler but nonoptimal control design for the sail control in the xz-plane of the problem, where the equilibria of practical interest are located.

Because the eccentricity of Earth's orbit is small, a simple feedback control law can be designed. First, we design the feedback gains for the system,

$$\chi' = A_0\chi + B_0\boldsymbol{u} \tag{26}$$

where matrices A_0 and B_0 are obtained by substituting $e = 0$ into matrices A and B of Eq. (23) and so full state feedback control is defined by

$$\boldsymbol{u} = G\chi \tag{27}$$

where the gain matrix G will be obtained by pole assignment. After a control scheme is designed, the stability of the controlled system must be numerically checked using Floquet theory as discussed in the preceding, because these gains are not exactly designed for the original system, defined by Eq. (23), but for the approximate system, defined by Eq. (26). An example is given to verify the possibility of solar sail control around an approximated equilibrium point in the xz plane of the sun–Earth elliptical restricted three-body problem. The required equilibrium point is defined as $x = 0.9$, $y = 0$, $z = 0.1$, which requires a sail lightness number $\beta = 0.304779$. If the desired poles are defined as $-5 \pm 2i$, $-5 \pm 3i$, $-6 \pm 3i$ the feedback gain matrix G is obtained as

$$G = \begin{bmatrix} 5.2520 & 0.0000 & 0.6402 & 2.1416 & 2.1416 & -0.4160 \\ 0.0000 & 13.6908 & -0.0000 & -5.8857 & 11.7714 & -0.0000 \\ -3.5342 & -0.0000 & -0.4522 & -1.6177 & -1.6177 & 5.7060 \end{bmatrix} \tag{28}$$

where the choice of the closed-loop poles is arbitrary, because the control design is not the main subject of this paper. Control is designed here to demonstrate the possibility of controlling a solar sail at an approximate equilibrium point. The behavior of the controlled time-varying system [Eq. (23)] with feedback gains G, which is obtained by the approximated system [Eq. (26)], is shown in Fig. 5. It can be seen that the out-of-plane equilibria are completely controlled by the linear feedback scheme in the sun–Earth elliptical restricted three-body problem.

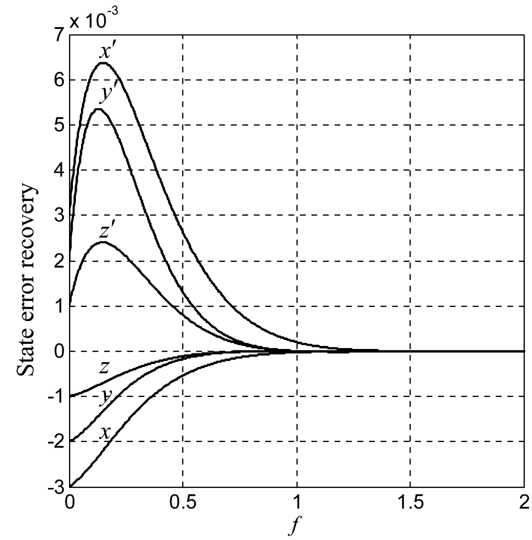

Fig. 5 Behavior of controlled time-varying linear system.

Conclusions

This paper has investigated the existence and dynamical properties of the equilibria of solar sails in the elliptical restricted three-body problem. The results show that, because of the stretching of the system plane coordinates and unstretching of the out-of-plane coordinate, there are no equilibrium surfaces in the three-dimensional, elliptical restricted three-body problem. But two-dimensional planar equilibrium curves still exist in the plane of the system, with curve trajectories in the nonuniformly rotating, pulsating system the same as in the circular case. Numerical analysis shows that these equilibrium points are unstable. Although the out-of-plane equilibrium points do not exist, this is problematic only for some proposed missions. When $e \ll 1$ the sail can be controlled around approximated equilibrium points. Therefore, solar sails can still be used for practical applications at out-of-plane equilibria of the sun–Earth elliptical restricted three-body problem.

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